

# Accuracy on eigenvalues for a Schrödinger operator with a degenerate potential in the semi-classical limit

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## Abstract

We consider a semi-classical Schrödinger operator  $-h^2\Delta + V$  with a degenerate potential  $V(x,y) = f(x)g(y)$ .  $g$  is assumed to be a homogeneous positive function of  $m$  variables and  $f$  is a strictly positive function of  $n$  variables, with a strict minimum. We give sharp asymptotic behaviour of low eigenvalues bounded by some power of the parameter  $h$ , by improving Born-Oppenheimer approximation.

## 1 Introduction

In our paper [MoTr] we have considered the Schrödinger operator on  $L^2(\mathbb{R}_x^n \times \mathbb{R}_y^m)$

$$\hat{H}_h = h^2 D_x^2 + h^2 D_y^2 + f(x)g(y) \quad (1.1)$$

with  $g \in C^\infty(\mathbb{R}^m \setminus \{0\})$  homogeneous of degree  $a > 0$ ,

$$g(\mu y) = \mu^a g(y) > 0, \quad \forall \mu > 0 \text{ and } \forall y \in \mathbb{R}^m \setminus \{0\}. \quad (1.2)$$

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<sup>1</sup>Keywords : eigenvalues, semi-classical asymptotics, Born-Oppenheimer approximation.

Mathematical Classification : 35P20.

$h > 0$  is a semiclassical parameter we assume to be small.

We have investigated the asymptotic behavior of the number of eigenvalues less than  $\lambda$  of  $\hat{H}_h$ ,

$$N(\lambda, \hat{H}_h) = \text{tr}(\chi_{]-\infty, \lambda[}(\hat{H}_h)) = \sum_{\lambda_k(\hat{H}_h) < \lambda} 1. \quad (1.3)$$

( $\text{tr}(P)$  denotes the trace of the operator  $P$ ).

If  $P$  is a self-adjoint operator on a Hilbert space  $\mathcal{H}$ , we denote respectively by  $sp(P)$ ,  $sp_{ess}(P)$  and  $sp_d(P)$  the spectrum, the essential spectrum and the discrete spectrum of  $P$ .

When  $-\infty < \inf sp(P) < \inf sp_{ess}(P)$ , we denote by  $(\lambda_k(P))_{k>0}$  the increasing sequence of eigenvalues of  $P$ , repeated according to their multiplicity:

$$sp_d(P) \cap ]-\infty, \inf sp_{ess}(P)[ = \{\lambda_k(P)\}. \quad (1.4)$$

In this paper we are interested in a sharp estimate for some eigenvalues of  $\hat{H}_h$ . We make the following assumptions on the other multiplicative part of the potential:

$$\begin{aligned} f &\in C^\infty(\mathbb{R}^n), \forall \alpha \in \mathbb{N}^n, (|f(x)| + 1)^{-1} \partial_x^\alpha f(x) \in L^\infty(\mathbb{R}^n) \\ 0 &< f(0) = \inf_{x \in \mathbb{R}^n} f(x) \\ f(0) &< \liminf_{|x| \rightarrow \infty} f(x) = f(\infty) \\ \partial^2 f(0) &> 0 \end{aligned} \quad (1.5)$$

$\partial^2 f(a)$  denotes the hessian matrix:

$$\partial^2 f(a) = \left( \frac{\partial^2}{\partial x_i \partial x_j} f(a) \right)_{1 \leq i, j \leq n}.$$

By dividing  $\hat{H}_h$  by  $f(0)$ , we can change the parameter  $h$  and assume that

$$f(0) = 1. \quad (1.6)$$

Let us define :  $\hbar = h^{2/(2+a)}$  and change  $y$  in  $y\hbar$ ; we can use the homogeneity of  $g$  (1.2) to get :

$$sp(\hat{H}_h) = \hbar^a sp(\hat{H}^\hbar), \quad (1.7)$$

with  $\hat{H}^{\hbar} = \hbar^2 D_x^2 + D_y^2 + f(x)g(y) = \hbar^2 D_x^2 + Q(x, y, D_y) :$

$$Q(x, y, D_y) = D_y^2 + f(x)g(y) .$$

Let us denote the increasing sequence of eigenvalues of  $D_y^2 + g(y)$ , (on  $L^2(\mathbb{R}^m)$ ), by  $(\mu_j)_{j>0}$ .

The associated eigenfunctions will be denoted by  $(\varphi_j)_j :$

$$\begin{aligned} D_y^2 \varphi_j(y) + g(y) \varphi_j(y) &= \mu_j \varphi_j(y) \\ \langle \varphi_j | \varphi_k \rangle &= \delta_{jk} \end{aligned} \quad (1.8)$$

and  $(\varphi_j)_j$  is a Hilbert base of  $L^2(\mathbb{R}^m)$ .

By homogeneity (1.2) the eigenvalues of  $Q_x(y, D_y) = D_y^2 + f(x)g(y)$ , on  $L^2(\mathbb{R}^m)$ , for a fixed  $x$ , are given by the sequence  $(\lambda_j(x))_{j>0}$ , where :  $\lambda_j(x) = \mu_j f^{2/(2+a)}(x)$ .

So as in [MoTr] we get :

$$\hat{H}^{\hbar} \geq [\hbar^2 D_x^2 + \mu_1 f^{2/(2+a)}(x)] . \quad (1.9)$$

This estimate is sharp as we will see below.

Then using the same kind of estimate as (1.9), one can see that

$$\inf \text{sp}_{ess}(\hat{H}^{\hbar}) \geq \mu_1 f^{2/(2+a)}(\infty) . \quad (1.10)$$

We are in the Born-Oppenheimer approximation situation described by A. Martinez in [Ma] : the "effective" potential is given by  $\lambda_1(x) = \mu_1 f^{2/(2+a)}(x)$ , the first eigenvalue of  $Q_x$ , and the assumptions on  $f$  ensure that this potential admits one unique and nondegenerate well  $U = \{0\}$ , with minimal value equal to  $\mu_1$ . Hence we can apply theorem 4.1 of [Ma] and get :

**Theorem 1.1** *Under the above assumptions, for any arbitrary  $C > 0$ , there exists  $h_0 > 0$  such that, if  $0 < \hbar < h_0$ , the operator  $(\hat{H}^{\hbar})$  admits a finite number of eigenvalues  $E_k(\hbar)$  in  $[\mu_1, \mu_1 + C\hbar]$ , equal to the number of the eigenvalues  $e_k$  of  $D_x^2 + \frac{\mu_1}{2+a} < \partial^2 f(0) x, x >$  in  $[0, +C]$  such that :*

$$E_k(\hbar) = \lambda_k(\hat{H}^{\hbar}) = \lambda_k(\hbar^2 D_x^2 + \mu_1 f^{2/(2+a)}(x)) + \mathbf{O}(\hbar^2) . \quad (1.11)$$

More precisely  $E_k(\hbar) = \lambda_k(\hat{H}^{\hbar})$  has an asymptotic expansion

$$E_k(\hbar) \sim \mu_1 + \hbar (e_k + \sum_{j \geq 1} \alpha_{kj} \hbar^{j/2}) . \quad (1.12)$$

If  $E_k(\hbar)$  is asymptotically non degenerated, then there exists a quasimode

$$\phi_k^\hbar(x, y) \sim \hbar^{-m_k} e^{-\psi(x)/\hbar} \sum_{j \geq 0} \hbar^{j/2} a_{kj}(x, y), \quad (1.13)$$

satisfying

$$\begin{aligned} C_0^{-1} &\leq \|\hbar^{-m_k} e^{-\psi(x)/\hbar} a_{k0}(x, y)\| \leq C_0 \\ \|\hbar^{-m_k} e^{-\psi(x)/\hbar} a_{kj}(x, y)\| &\leq C_j \\ \left\| \left( \widehat{H}^\hbar - \mu_1 - \hbar e_k - \sum_{1 \leq j \leq J} \alpha_{kj} \hbar^{j/2} \right) \right. \\ &\left. \hbar^{-m_k} e^{-\psi(x)/\hbar} \sum_{0 \leq j \leq J} \hbar^{j/2} a_{kj}(x, y) \right\| \leq C_J \hbar^{(J+1)/2} \end{aligned} \quad (1.14)$$

The formula (1.12) implies

$$\lambda_k(\widehat{H}^\hbar) = \mu_1 + \hbar \lambda_k \left( D_x^2 + \frac{\mu_1}{2+a} < \partial^2 f(0) x, x > \right) + \mathbf{O}(\hbar^{3/2}), \quad (1.15)$$

and when  $k = 1$ , one can improve  $\mathbf{O}(\hbar^{3/2})$  into  $\mathbf{O}(\hbar^2)$ . The function  $\psi$  is defined by :  $\psi(x) = d(x, 0)$ , where  $d$  denotes the Agmon distance related to the degenerate metric  $\mu_1 f^{2/(2+a)}(x) dx^2$ .

## 2 Lower energies

We are interested now with the lower energies of  $\widehat{H}^\hbar$ . Let us make the change of variables

$$(x, y) \rightarrow (x, f^{1/(2+a)}(x)y). \quad (2.1)$$

The Jacobian of this diffeomorphism is  $f^{m/(2+a)}(x)$ , so we perform the change of test functions :  $u \rightarrow f^{-m/(4+2a)}(x)u$ , to get a unitary transformation.

Thus we get that

$$sp(\widehat{H}^\hbar) = sp(\widetilde{H}^\hbar) \quad (2.2)$$

where  $\widetilde{H}^\hbar$  is the self-adjoint operator on  $L^2(\mathbb{R}^n \times \mathbb{R}^m)$  given by

$$\widetilde{H}^\hbar = \hbar^2 L^*(x, y, D_x, D_y) L(x, y, D_x, D_y) + f^{2/(2+a)}(x) (D_y^2 + g(y)), \quad (2.3)$$

with

$$L(x, y, D_x, D_y) = D_x + \frac{1}{(2+a)f(x)} [(y D_y) - i \frac{m}{2}] \nabla f(x).$$

We decompose  $\tilde{H}^{\hbar}$  in four parts :

$$\begin{aligned}
\tilde{H}^{\hbar} &= \hbar^2 D_x^2 + f^{2/(2+a)}(x) (D_y^2 + g(y)) \\
&+ \hbar^2 \frac{2}{(2+a)f(x)} (\nabla f(x) D_x)(y D_y) \\
&+ i \hbar^2 \frac{1}{(2+a)f^2(x)} (|\nabla f(x)|^2 - f(x) \Delta f(x)) [(y D_y) - i \frac{m}{2}] \\
&+ \hbar^2 \frac{1}{(2+a)^2 f^2(x)} |\nabla f(x)|^2 [(y D_y)^2 + \frac{m^2}{4}]
\end{aligned} \tag{2.4}$$

Our goal is to prove that the only significant role up to order 2 in  $\hbar$  will be played by the first operator, namely :  $\tilde{H}_1^{\hbar} = \hbar^2 D_x^2 + f^{2/(2+a)}(x) (D_y^2 + g(y))$ .

Let us denote by  $\nu_{j,k}^{\hbar}$  the eigenvalues of the operator  $\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x)$  and by  $\psi_{j,k}^{\hbar}$  the associated normalized eigenfunctions .

Let us consider the following test functions :

$$u_{j,k}^{\hbar}(x, y) = \psi_{j,k}^{\hbar}(x) \varphi_j(y) ,$$

where the  $\varphi_j$ 's are the eigenfunctions defined in (1.8); we have immediately :

$$\tilde{H}_1^{\hbar}(u_{j,k}^{\hbar}(x, y)) = \nu_{j,k}^{\hbar} u_{j,k}^{\hbar}(x, y) .$$

We will need the following lemma :

**Lemma 2.1** . *For any integer  $N$  , there exists a positive constant  $C$  depending only on  $N$  such that for any  $k \leq N$  , the eigenfunction  $\psi_{j,k}^{\hbar}$  satisfies the following inequalities : for any  $\alpha \in \mathbb{N}^n$  ,  $|\alpha| \leq 2$  ,*

$$\begin{aligned}
\| \hbar_j^{|\alpha|/2} |D_x^{\alpha} \psi_{j,k}^{\hbar}| \| &< C \\
\| \left( \frac{\nabla f(x)}{f(x)} \right)^{\alpha} \psi_{j,k}^{\hbar} \| &< \hbar_j^{|\alpha|/2} C
\end{aligned} \tag{2.5}$$

with  $\hbar_j = \hbar \mu_j^{-1/2}$  .

**Proof.**

Let us recall that it is well known, (see [He-Sj1] ), that

$$\forall k \leq N , \quad \mu_j^{-1} \nu_{j,k}^{\hbar} = 1 + \mathbf{O}(\hbar_j) .$$

It is clear also that

$$[\hbar_j^2 D_x^2 + f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}] \psi_{j,k}^{\hbar}(x) = 0 . \tag{2.6}$$

We shall need the following inequality, that we can derive easily from (2.6) and the Agmon estimate (see [He-Sj1]) :  $\forall \varepsilon \in ]0, 1[$ ,

$$\begin{aligned} \varepsilon \int [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}]_+ e^{2(1-\varepsilon)^{1/2} d_{j,k}(x)/\hbar_j} |\psi_{j,k}^{\hbar}(x)|^2 dx \leq \\ \int [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}]_- |\psi_{j,k}^{\hbar}(x)|^2 dx, \end{aligned} \quad (2.7)$$

where  $d_{j,k}$  is the Agmon distance associated to the metric  $[f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}]_+ dx^2$ .

Let us prove the lemma for  $|\alpha| = 1$ .

As  $\int [\hbar_j^2 |D_x \psi_{j,k}^{\hbar}(x)|^2 + (f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}) |\psi_{j,k}^{\hbar}(x)|^2] dx = 0$ ,  
 $\mu_j^{-1} \nu_{j,k}^{\hbar} - 1 = \mathbf{O}(\hbar_j)$ , and  $f^{2/(2+a)}(x) - 1 > 0$ ,  
 we get that  $\hbar_j \| |D_x \psi_{j,k}^{\hbar}(x)| \|^2 \leq C$ .

Furthermore, we use that  $C^{-1} |\nabla f(x)|^2 \leq f^{2/(2+a)}(x) - 1 \leq C |\nabla f(x)|^2$ ,  
 for  $|x| \leq C^{-1}$ , the exponential decreasing (in  $\hbar_j$ ) of  $\psi_{j,k}^{\hbar}$  given by (2.7)  
 and the boundness of  $|\nabla f(x)|/f(x)$  to get

$$\left\| \frac{|\nabla f(x)|}{f(x)} \psi_{j,k}^{\hbar}(x) \right\|^2 \leq C \int [f^{2/(2+a)}(x) - 1] |\psi_{j,k}^{\hbar}(x)|^2 dx \leq \hbar_j C.$$

Now we study the case  $|\alpha| = 2$ .

If  $c_0 \in ]0, 1]$  is large enough and  $|x| \in [\hbar_j^{1/2} c_0, 2c_0]$ , then we have

$$|x|^2/C \leq f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar} \leq C|x|^2 \quad (2.8)$$

Therefore there exists  $C_1 > 1$  such that  $C_1^{-1}|x|^2 \leq d_{j,k}(x) \leq C_1|x|^2$ ,  
 and then

$$|x|^2 \leq \hbar_j C e^{d_{j,k}(x)/\hbar_j}. \quad (2.9)$$

Then the inequality :  $C^{-1}|x| \leq |\nabla f(x)| \leq C|x|$  together with (2.8),  
 (2.9) and (2.7) entail that

$$\begin{aligned} \int_{|x| \geq C_0 \hbar_j^{1/2}} \frac{|\nabla f(x)|^4}{f^4(x)} |\psi_{j,k}^{\hbar}(x)|^2 dx \\ \leq \hbar_j C \int [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}]_+ e^{d_{j,k}(x)/\hbar_j} |\psi_{j,k}^{\hbar}(x)|^2 dx \\ \leq \hbar_j C \int [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}]_- |\psi_{j,k}^{\hbar}(x)|^2 dx \\ \leq \hbar_j^2 C. \end{aligned}$$

It remains to estimate  $\hbar_j^2 \|D_x^\alpha \psi_{j,k}^{\hbar}(x)\|$  with  $|\alpha| = 2$ .

We use that  $-\hbar_j^2 \Delta \psi_{j,k}^{\hbar}(x) = [-f^{2/(2+a)}(x) + \mu_j^{-1} \nu_{j,k}^{\hbar}] \psi_{j,k}^{\hbar}(x)$ ,  
 and that we have proved that  $\|[-f^{2/(2+a)}(x) + \mu_j^{-1} \nu_{j,k}^{\hbar}] \psi_{j,k}^{\hbar}(x)\| \leq \hbar_j C$  ;  
 so  $\|D_x^\alpha \psi_{j,k}^{\hbar}(x)\| \leq C/\hbar_j$  if  $|\alpha| = 2$ .

We will need the following result.

**Proposition 2.2** *Let  $V(y) \in C^\infty(\mathbb{R}^m)$  such that*

$$\begin{aligned} \exists s > 0, C_0 > 0 \text{ s.t. } -C_0 + |y|^s/C_0 \leq V(y) \leq C_0(|y|^s + 1) \\ \forall \alpha \in \mathbb{N}^m, (1 + |y|^2)^{(s-|\alpha|)/2} \partial_y^\alpha V(y) \in L^\infty(\mathbb{R}^m). \end{aligned} \quad (2.10)$$

*If  $u(y) \in L^2(\mathbb{R}^m)$  and  $D_y^2 u(y) + V(y)u(y) \in S(\mathbb{R}^m)$ , then  $u \in S(\mathbb{R}^m)$ . ( $S(\mathbb{R}^m)$  is the Schwartz space).*

The proof comes from the fact that there exists a parametrix of  $D_y^2 + V(y)$  in some class of pseudodifferential operator: see for the more general case in [Hor], or for this special case in Shubin book [Shu].

**Theorem 2.3 .**

*Under the assumptions (1.2) and (1.5), for any fixed integer  $N > 0$ , there exists a positive constant  $h_0(N)$  verifying : for any  $\hbar \in ]0, h_0(N)[$ , for any  $k \leq N$  and any  $j \leq N$  such that*

$$\mu_j < \mu_1 f^{2/(2+a)}(\infty),$$

*there exists an eigenvalue  $\lambda_{jk} \in \text{sp}_d(\hat{H}^\hbar)$  such that*

$$| \lambda_{jk} - \lambda_k(\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x)) | \leq \hbar^2 C. \quad (2.11)$$

*Consequently, when  $k = 1$ , we have*

$$| \lambda_{j1} - \left[ \mu_j + \hbar(\mu_j)^{1/2} \frac{\text{tr}((\partial^2 f(0))^{1/2})}{(2+a)^{1/2}} \right] | \leq \hbar^2 C. \quad (2.12)$$

**Proof .**

The first part of the theorem will follow if we prove that :

$$\|(\hat{H}^\hbar - \tilde{H}_1^\hbar)(u_{j,k}^\hbar(x, y))\| = \|(\hat{H}^\hbar - \nu_{j,k}^\hbar)u_{j,k}^\hbar(x, y)\| = \mathbf{O}(\hbar^2).$$

Let us consider a function  $\chi \in C^\infty(\mathbb{R})$  such that

$$\begin{aligned} \chi(t) &= 1 \text{ if } |t| \leq 1/2 \text{ and} \\ \chi(t) &= 0 \text{ if } |t| > 1. \end{aligned}$$

Then  $(D_y^2 + g(y))(1 - \chi(|y|))\varphi_j(y) \in S(\mathbb{R}^m)$ , and Proposition 2.2 shows that  $(1 - \chi(|y|))\varphi_j(y) \in S(\mathbb{R}^m)$ . As  $D_y^2 \varphi_j(y) = (\mu_j - g(y))\varphi_j(y)$ , we get that

$$\forall k \in \mathbb{N}, (1 + |y|)^k [|\varphi_j(y)|^2 + |D_y \varphi_j(y)|^2 + |D_y^2 \varphi_j(y)|^2] \in L^1(\mathbb{R}^m). \quad (2.13)$$

The quantity  $(\widehat{H}^{\hbar} - \widetilde{H}_1^{\hbar})(u_{j,k}^{\hbar}(x,y))$  is, by (2.4), composed of 3 parts. According to Lemma 2.1 and the estimate (2.13), the two last parts are bounded in  $L^2$ -norm by  $\hbar^2 C$ , ( $\mu_j \leq C$ ).

To obtain a bound for the first part, we integrate by parts to get that

$$\left\| \frac{\nabla f(x)}{f(x)} D_x \psi_{j,k}^{\hbar} \right\|^2 \leq C \left[ \|D_x^2 \psi_{j,k}^{\hbar}\| \times \left\| \frac{|\nabla f(x)|^2}{f^2(x)} \psi_{j,k}^{\hbar} \right\| + \|D_x \psi_{j,k}^{\hbar}\| \times \left\| \frac{|\nabla f(x)|}{f(x)} \psi_{j,k}^{\hbar} \right\| \right],$$

and then we use again Lemma 2.1. Thus :  $\left\| \frac{\nabla f(x)}{f(x)} D_x \psi_{j,k}^{\hbar} \right\| \leq C$ .

According to estimate (2.13) we have finally  $\left\| \frac{\nabla f(x)}{f(x)} D_x (y D_y) u_{j,k}^{\hbar} \right\| \leq C$ .

### 3 Middle energies

We are going to refine the preceding results when  $a \geq 2$  and  $f(\infty) = \infty$ . It is possible then to get sharp localization near the  $\mu_j$ 's for much higher values of  $j$ 's. More precisely we prove :

**Theorem 3.1** . *We assume (1.5) with  $f(\infty) = \infty$ , (1.2) with  $a \geq 2$  and with  $g \in C^\infty(\mathbb{R}^m)$ .*

*Let us consider  $j$  such that  $\mu_j \leq \hbar^{-2}$  ; then for any integer  $N$ , there exists a constant  $C$  depending only on  $N$  such that, for any  $k \leq N$ , there exists an eigenvalue  $\lambda_{jk} \in \text{sp}_d(\widehat{H}^{\hbar})$  verifying*

$$|\lambda_{jk} - \lambda_k(\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x))| \leq C \mu_j \hbar^2. \quad (3.1)$$

*Consequently, when  $k = 1$ , we have*

$$|\lambda_{j1} - \left[ \mu_j + \hbar(\mu_j)^{1/2} \frac{\text{tr}((\partial^2 f(0))^{1/2})}{(2+a)^{1/2}} \right]| \leq C \mu_j \hbar^2. \quad (3.2)$$

**Proof :**

Let us define the class of symbols  $S(p^s(y, \eta))$ ,  $s \in \mathbb{R}$ , with  $p(y, \eta) = |\eta|^2 + g(y) + 1$ .

$$q(y, \eta) \in S(p^s(y, \eta)) \quad \text{iff} \quad q(y, \eta) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$$

and for any  $\alpha$  and  $\beta \in \mathbb{N}^m$ ,

$$p^{-s}(y, \eta)(|\eta| + 1)^{-|\alpha|}(|y| + 1)^{-|\beta|} D_\eta^\alpha D_y^\beta q(y, \eta) \in L^\infty(\mathbb{R}^{2m}).$$



For such a symbol  $q(y, \eta) \in S(p^s(y, \eta))$ , we define the operator  $Q$  on  $S(\mathbb{R}^m)$  :

$$Qf(y) = (2\pi)^{-m} \int_{\mathbb{R}^{2m}} q\left(\frac{y+z}{2}, \eta\right) e^{i(y-z)\eta} f(z) dz d\eta .$$

We will say that  $Q \in OPS(p^s(y, \eta))$ .

It is well known, (see [Hor]) that  $(D_y^2 + g(y))^s \in OPS(p^s(y, \eta))$ .

As  $a \geq 2$ , we get that  $yD_y \in OPS(p(y, \eta))$ , and then that  $yD_y(D_y^2 + g(y))^{-1} \in OPS(1)$ .

Therefore  $yD_y(D_y^2 + g(y))^{-1}$  and  $(yD_y)^2(D_y^2 + g(y))^{-2}$  are bounded operator on  $L^2(\mathbb{R}^m)$ , and we get as a consequence the following bound :

$$\mu_j^{-1} \|yD_y \varphi_j\| + \mu_j^{-2} \|(yD_y)^2 \varphi_j\| \leq C . \quad (3.3)$$

As in the proof of Theorem 2.3, using (3.3) instead of (2.13), we get easily that

$$\|(\hat{H}^{\hbar} - \tilde{H}^{\hbar})u_{j,k}^{\hbar}\| \leq C[\hbar^2 \mu_j + \hbar^3 \mu_j^{3/2}] \leq C\hbar^2 \mu_j ,$$

and then Theorem 3.1 follows.

## 4 An application

We consider a Schrödinger operator on  $L^2(\mathbb{R}_z^d)$  with  $d \geq 2$ ,

$$P^{\hbar} = -\hbar^2 \Delta + V(z) \quad (4.1)$$

with a real and regular potential  $V(z)$  satisfying

$$\begin{aligned} V &\in C^\infty(\mathbb{R}^d; [0, +\infty[) \\ \liminf_{|z| \rightarrow \infty} V(z) &> 0 \\ \Gamma = V^{-1}(\{0\}) &\text{ is a regular hypersurface.} \end{aligned} \quad (4.2)$$

By hypersurface, we mean a submanifold of codimension 1. Moreover we assume that  $\Gamma$  is connected and that there exist  $m \in \mathbb{N}^*$  and  $C_0 > 0$  such that for any  $z$  verifying  $d(z, \Gamma) < C_0^{-1}$

$$C_0^{-1} d^{2m}(z, \Gamma) \leq V(z) \leq C_0 d^{2m}(z, \Gamma) \quad (4.3)$$

( $d(E, F)$  denotes the euclidian distance between  $E$  and  $F$ ).

We choose an orientation on  $\Gamma$  and a unit normal vector  $N(s)$  on each  $s \in \Gamma$ , and then, we can define the function on  $\Gamma$ ,

$$f(s) = \frac{1}{(2m)!} \left( N(s) \frac{\partial}{\partial s} \right)^{2m} V(s), \quad \forall s \in \Gamma. \quad (4.4)$$

Then by (4.2) and (4.6),  $f(s) > 0$ ,  $\forall s \in \Gamma$ .

Finally we assume that the function  $f$  achieves its minimum on  $\Gamma$  on a finite number of discrete points:

$$\Sigma_0 = f^{-1}(\{\eta_0\}) = \{s_1, \dots, s_{\ell_0}\}, \quad \text{if } \eta_0 = \min_{s \in \Gamma} f(s), \quad (4.5)$$

and the hessian of  $f$  at each point  $s_j \in \Sigma_0$  is non degenerated:

$$\exists \eta_1 > 0 \text{ s.t.}$$

$$\frac{1}{2} \langle d(\langle df; w \rangle); w \rangle(s_j) \geq \eta_1 |w(s_j)|^2, \quad \forall w \in T\Gamma, \forall s_j \in \Sigma_0. \quad (4.6)$$

If  $g = (g_{ij})$  is the riemannian metric on  $\Gamma$ , then  $|w(s)| = (g(w(s), w(s)))^{1/2}$ . The hessian of  $f$  at each  $s_j \in \Sigma_0$ , is the symmetric operator on  $T_{s_j}\Gamma$ ,  $Hess(f)_{s_j}$ , associated to the two-bilinear form defined on  $T_{s_j}\Gamma$  by :

$$(v, w) \in (T_{s_j}\Gamma)^2 \rightarrow \frac{1}{2} \langle d(\langle df; \tilde{v} \rangle); \tilde{w} \rangle(s_j), \quad (4.7)$$

$$\forall (\tilde{v}, \tilde{w}) \in (T\Gamma)^2 \text{ s.t. } (\tilde{v}(s_j), \tilde{w}(s_j)) = (v, w).$$

$Hess(f)_{s_j}$  has  $d-1$  non negative eigenvalues

$$\rho_1^2(s_j) \leq \dots \leq \rho_{d-1}^2(s_j), \quad (\rho_j(s_j) > 0).$$

In local coordinates, those eigenvalues are the ones of the symmetric matrix

$$\frac{1}{2} G^{1/2}(s_j) \left( \frac{\partial^2}{\partial x_k \partial x_\ell} f(s_j) \right)_{1 \leq k, \ell \leq d-1} G^{1/2}(s_j), \quad (G(x) = (g_{k,\ell}(x))_{1 \leq k, \ell \leq d-1}).$$

The eigenvalues  $\rho_k^2(s_j)$  do not depend on the choice of coordinates. We denote

$$Tr^+(Hess(f(s_j))) = \sum_{\ell=1}^{d-1} \rho_\ell(s_j). \quad (4.8)$$

We denote by  $(\mu_j)_{j \geq 1}$  the increasing sequence of the eigenvalues of the operator  $-\frac{d^2}{dt^2} + t^{2m}$  on  $L^2(\mathbb{R})$ , and by  $(\varphi_j(t))_{j \geq 1}$  the associated orthonormal Hilbert base of eigenfunctions.

**Theorem 4.1** *Under the above assumptions, for any  $N \in \mathbb{N}^*$ , there exist  $h_0 \in ]0, 1]$  and  $C_0 > 0$  such that, if  $\mu_j \ll h^{-4m/(m+1)(2m+3)}$ , and if  $\alpha \in \mathbb{N}^{d-1}$  and  $|\alpha| \leq N$ , then  $\forall s_\ell \in \Sigma_0$ ,  $\exists \lambda_{j\ell\alpha}^h \in sp_d(P^h)$  s.t.*

$$\left| \lambda_{j\ell\alpha}^h - h^{2m/(m+1)} \left[ \eta_0^{1/(m+1)} \mu_j + h^{1/(m+1)} \mu_j^{1/2} \mathcal{A}_\ell(\alpha) \right] \right| \leq h^2 \mu_j^{2+3/2m} C_0 ;$$

$$\text{with } \mathcal{A}_\ell(\alpha) = \frac{1}{\eta_0^{m/(2m+2)} (m+1)^{1/2}} [2\alpha \rho(s_\ell) + \text{Tr}^+(Hess(f(s_\ell)))] .$$

$$(\alpha \rho(s_\ell) = \alpha_1 \rho_1(s_\ell) + \dots \alpha_{d-1} \rho_{d-1}(s_\ell) ) .$$

**Proof :**

Let  $\mathcal{O}_0 \subset \mathbb{R}^d$  be an open neighbourhood of  $s_l \in \Sigma_0$ , such that there exists  $\phi \in C^\infty(\mathcal{O}_0; \mathbb{R})$  satisfying

$$\begin{aligned} \Gamma_0 &= \Gamma \cap \mathcal{O}_0 = \{z \in \mathcal{O}_0; \phi(z) = 0\} ; \\ |\nabla \phi(z)| &= 1, \quad \forall z \in \mathcal{O}_0 . \end{aligned} \quad (4.9)$$

After changing  $\mathcal{O}_0$  into a smaller neighbourhood if necessary, we can find  $\tau \in C^\infty(\mathcal{O}_0; \mathbb{R}^{d-1})$  such that  $\tau(s_l) = 0$  and  $\forall z \in \mathcal{O}_0$ ,

$$\begin{aligned} \nabla \tau_j(z) \cdot \nabla \phi(z) &= 0, \quad \forall j = 1, \dots, d-1 \\ \text{rank}\{\nabla \tau_1(z), \dots, \nabla \tau_{d-1}(z)\} &= d-1 . \end{aligned} \quad (4.10)$$

Then  $(x, y) = (x_1, \dots, x_{d-1}, y) = (\tau_1, \dots, \tau_{d-1}, \phi)$  are local coordinates in  $\mathcal{O}_0$  such that

$$\begin{aligned} \Delta &= |\tilde{g}|^{-1/2} \sum_{1 \leq i, j \leq d-1} \partial_{x_i} (|\tilde{g}|^{1/2} \tilde{g}^{ij} \partial_{x_j}) + |\tilde{g}|^{-1/2} \partial_y (|\tilde{g}|^{1/2} \partial_y) \\ V &= y^{2m} f(x, y) \quad \text{with } f \in C^\infty(\mathcal{V}_0) ; \end{aligned} \quad (4.11)$$

$\mathcal{V}_0$  is an open neighbourhood of zero in  $\mathbb{R}^d$ ,  
 $\tilde{g}^{ij}(x, y) = \tilde{g}^{ji}(x, y) \in C^\infty(\mathcal{V}_0; \mathbb{R})$ ,  $|\tilde{g}|^{-1} = \det(\tilde{g}^{ij}(x, y)) > 0$ .  
 $x = (x_1, \dots, x_{d-1})$  are local coordinates on  $\Gamma_0$   
and the metric  $g = (g_{ij})$  on  $\Gamma_0$  is given by

$$(g_{ij}(x))_{1 \leq i, j \leq d-1} = G(x), \quad \text{with } (G(x))^{-1} = (\tilde{g}^{ij}(x, 0))_{1 \leq i, j \leq d-1} .$$

If  $w \in C_0^2(\mathcal{O}_0)$  then

$$\begin{aligned} P^h w &= \widehat{P}^h u \quad \text{with} \\ u &= |\widetilde{g}|^{1/4} w \quad \text{and} \\ \widehat{P}^h &= -h^2 \sum_{1 \leq i, j \leq d-1} \partial_{x_i} (\widetilde{g}^{ij} \partial_{x_j}) - h^2 \partial_y^2 + V + h^2 V_0, \end{aligned} \quad (4.12)$$

for some  $V_0 \in C^\infty(\mathcal{V}_0; \mathbb{R})$ .

Let us write

$$V(x, y) = y^{2m} f(x) + y^{2m+1} f_1(x) + y^{2m+2} \widetilde{f}_2(x, y) : \quad (4.13)$$

$f(x) = \widetilde{f}(x, 0)$  and  $\widetilde{f}_2 \in C^\infty(\mathcal{V}_0)$ .

We perform the change of variable (2.1) and the related unitary transformation,

$$(x, y) \rightarrow (x, t) = (x, f^{1/(2(m+1))}(x)y), \quad u \rightarrow v = f^{-1/(4(m+1))}u,$$

to get that

$$\begin{aligned} \widehat{P}^h u &= \widehat{Q}^h v \quad \text{with} \\ \widehat{Q}^h &= Q_0^h + t^{2m+1} f_1^0(x) + h^2 R_0 + h^2 t R_1 + t^{2m+2} \widetilde{f}_2^0 : \\ Q_0^h &= -h^2 \sum_{1 \leq i, j \leq d-1} \partial_{x_i} (g^{ij} \partial_{x_j}) + f^{1/(m+1)}(x) (-h^2 \partial_t^2 + t^{2m}) \end{aligned} \quad (4.14)$$

and  $R_0 = ta(x, t)(\partial_x f(x) \partial_x) \partial_t + b(x, t) t \partial_t +$

$$\sum_{ij} b_{ij}(x, t) \partial_{x_i} f(x) \partial_{x_j} f(x) (t \partial_t)^2 + c(x, t),$$

$R_1 = \sum_{1 \leq i, j \leq d-1} \partial_{x_i} (\alpha_{ij}(x, t) \partial_{x_j})$ , all coefficients are regular in a neighbourhood of the zero in  $\mathbb{R}^d$ .

Let  $\mu_j$  be as in the theorem 4.1. We define  $h_j = h^{1/(m+1)}/\mu_j^{1/2}$ . Let  $\mathcal{O}'_0$  be a bounded open neighbourhood of zero in  $\mathbb{R}^{d-1}$  such that  $\overline{\mathcal{O}'_0} \subset \mathcal{O}_0 \cap \{(x, 0); x \in \mathbb{R}^{d-1}\}$ .

We consider the Dirichlet operator on  $L^2(\mathcal{O}'_0)$ ,  $H_0^{h_j}$  :

$$H_0^{h_j} = -h_j^2 \sum_{1 \leq k, \ell \leq d-1} \partial_{x_k} (g^{k\ell}(x) \partial_{x_\ell}) + f^{1/(m+1)}(x). \quad (4.15)$$

It is well known, (see for example [He1] or [He-Sj1], that for any  $\alpha \in \mathbb{N}^{d-1}$  satisfying the assumptions of the theorem 4.1, one has:

$$\exists \lambda_{j,\alpha}^h \in sp(H_0^{h_j}) \quad \text{s.t.} \quad |\lambda_{j,\alpha}^h - [\eta_0^{1/(m+1)} + h_j \mathcal{A}_l(\alpha)]| \leq h_j^2 C ;$$

$\mathcal{A}_l(\alpha)$  is defined in theorem 4.1 in relation with our  $s_l \in \Sigma_0$ .

$C$  is a constant depending only on  $N$ . We will denote by  $\psi_{j,\alpha}^{h_j}(x)$  any associated eigenfunction with a  $L^2$ -norm equal to 1. Let  $\chi_0 \in C^\infty(\mathbb{R})$  such that

$$\chi_0(t) = 1 \quad \text{if} \quad |t| \leq 1/2 \quad \text{and} \quad \chi(t) = 0 \quad \text{if} \quad |t| \geq 1.$$

We define the following function :

$$u_{j,\alpha}^h(x, t) = h^{-1/(2m+2)} \chi_0(t/\epsilon_0) \psi_{j,\alpha}^{h_j}(x) [\varphi_j(h^{-1/(m+1)} t) - h^{1/(m+1)} F_j^h(x, t)] ,$$

with

$$F_j^h(x, t) = f_1^0(x) f^{-1/(m+1)}(x) \phi_j(h^{-1/(m+1)} t),$$

where  $\phi_j \in S(\mathbb{R})$  is solution of :

$$-\frac{d^2}{dt^2} \phi_j(t) + (t^{2m} - \mu_j) \phi_j(t) = t^{2m+1} \varphi_j(t) ,$$

and  $\epsilon_0 \in ]0, 1]$  is a small enough constant, but independent of  $h$  and  $j$ .  $\phi_j$  exists because  $\mu_j$  is a non-degenerated eigenvalue and the related eigenfunction  $\varphi_j$  (see 1.8) verifies  $\int_{\mathbb{R}} t^{2m+1} \varphi_j^2(t) dt = 0$ , since it is a real even or odd function.

Using the similar estimates as in chapter 3, one can get easily that

$$\mu_j^{-1} \|t \partial_t \varphi_j\|_{L^2(\mathbb{R})} + \mu_j^{-2} \|(t \partial_t)^2 \varphi_j\|_{L^2(\mathbb{R})} \leq C$$

and  $\forall k \in \mathbb{N}$ ,  $\exists C_k > 0$  s.t.  $\mu_j^{-k/2m} \|t^k \varphi_j\|_{L^2(\mathbb{R})} \leq C_k$ .

It is well known that there exists  $\epsilon_1 > 0$  s.t.

$|\mu_j - \mu_\ell| \geq \epsilon_1$ ,  $\forall \ell \neq j$ , then the inverse of  $-\frac{d^2}{dt^2} + t^{2m} - \mu_j$  is  $L^2(\mathbb{R})$ -bounded by  $1/\epsilon_1$ , (on the orthogonal of  $\varphi_j$ ). So in the same way as in chapter 3, we get also that

$$\mu_j^{-2-1/2m} \|t \partial_t \phi_j\|_{L^2(\mathbb{R})} + \mu_j^{-3-1/2m} \|(t \partial_t)^2 \phi_j\|_{L^2(\mathbb{R})} \leq C$$

and  $\forall k \in \mathbb{N}$ ,  $\exists C_k > 0$  s.t.  $\mu_j^{-1-(k+1)/2m} \|t^k \phi_j\|_{L^2(\mathbb{R})} \leq C_k$ .

As in the proof of Theorem 3.1, we get easily that

$$\|[\widehat{Q}^h - \mu_j \lambda_{j,\alpha}^h] \chi_0(|x|/\epsilon_0) u_{j,\alpha}^{h_j}(x, t)\|_{L^2(\mathcal{O}_0)} \leq h^2 \mu_j^{(4m+3)/2m} C$$

and

$$| \|\chi_0(|x|/\epsilon_0)u_{j,\alpha}^{h_j}(x,t)\|_{L^2(\mathcal{O}_0)} - 1 | = \mathbf{O}(h^{1/(m+1)}\mu_j^{(2m+1)/2m}) = o(1) .$$

So the theorem 4.1 follows easily.

**Remark 4.2** *If in Theorem 4.1 we assume that  $j$  is also bounded by  $N$ , then, as in [He-Sj4], we can get a full asymptotic expansion*

$$\lambda_{j\ell\alpha}^h \sim h^{2m/(m+1)} \sum_{k=0}^{+\infty} c_{j\ell k\alpha} h^{k/(m+1)} ,$$

and for the related eigenfunction, a quasimode of the form

$$u_{j\ell\alpha}^h(x,t) \sim c(h)e^{-\psi(x)/h^{1/(m+1)}} \chi_0(t/\epsilon_0) \sum_{k=0}^{+\infty} h^{k/(2m+2)} a_{j\ell k\alpha}(x) \phi_{jk}(t/h^{1/(m+1)}) .$$

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